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ORIGINAL PAPER

Dissipative operators with impulsive conditions

Ekin Uğurlu · Elgiz Bairamov

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Abstract In this paper a singular dissipative impulsive boundary value problem with n-impulsive points is investigated. In particular, using the Lidskiĭ's theorem it is proved that all eigen and associated functions of this problem is complete in the Hilbert space.

Keywords Dissipative operators · Impulsive conditions · Lidskiĭ's theorem

Mathematics Subject Classification (2000) 34B20 · 34B24 · 34B37

1 Introduction

Impulsive differential operators, that is, differential operators involving impulsive effects, appear as a natural description of observed evolution phenomena of several real world problems. Many physical, chemical, biological phenomena involving thresholds, bursting rhythm models in medicine, pharmacokinetics and frequency modulated systems and mathematical models in economics, do exhibit impulsive effects [1]. The theory of impulsive differential operators is a new and important branch of operator theory, which has an extensive physical, chemical and realistic mathematical model and hence has been emerging as an important area of investigation. Operator theory is useful to investigate the boundary value and impulsive boundary value problems (IBVPs). In these problems real-valued coefficients in the differential expressions and real parameters in the boundary-impulsive conditions generate the selfadjoint (symmetric) operators. It is well known that all eigenvalues are real of such operators.

E. Uğurlu (✉) · E. Bairamov
Department of Mathematics, Ankara University, 06100 Tandoğan, Ankara, Turkey
e-mail: ekinugurlu@yahoo.com

E. Bairamov
e-mail: bairamov@science.ankara.edu.tr

There are a lot of works about the spectral analysis of selfadjoint operators [2–9]. On the other hand non-real coefficients or non-real parameters generate nonselfadjoint operators. A kind of such operators have been investigated in [10–14].

An important part of nonselfadjoint operators is the class of dissipative operators. It is well known that all eigenvalues of the dissipative operators lie in the closed upper half-plane. This part only of the spectral analysis is so weak. Because distrubitions and multiplities of eigenvalues are not known with this analysis. Further another question is about whether the system of all eigen and associated functions associated with these eigenvalues span the space or not. To answer these questions there are some methods. Some of them are Livšic's theorem [13], Krein's theorem [13], Nagy–Foiaş's theorem [15] and Lidskiĭ's theorem [13]. The first three theorems were used in the literature to investigate the spectral analysis of the boundary value and IBVPs [16–24]. In this paper, the IBVP (3.1)–(3.5) with n-impulsive points ($2 \le n < \infty$) is investigated. In particular, using the Lidskiĭ's theorem it is proved that all eigen and associated functions of this problem is complete in the Hilbert space $L_w^2(\Lambda)$, where $\Lambda = \cup_{k=1}^{n+1} \Lambda_k$ and $\Lambda_k = (c_{k-1}, c_k)$.

2 Preliminaries

Let L denotes the linear nonselfadjoint operator in the Hilbert space H with the domain $D(L)$. The element $y \in D(L)$, $y \neq 0$, is called a root function of the operator L corresponding to the eigenvalue λ_0, if all powers of L are defined on this element and $(L - \lambda_0 I)^n y = 0$ for some $n > 0$. The set of all root functions of L corresponding to the eigenvalue λ_0 with $y = 0$ forms a linear set N_{λ_0} and is called the root lineal. The dimension of the lineal N_{λ_0} is called the algebraic multiplicity of the eigenvalue λ_0.

The functions $y_1, y_2, \ldots, y_k$ are called the associated functions of the eigenfunction y_0 if they belong to $D(L)$ and the equalities $Ly_j = \lambda_0 y_j + y_{j-1}$, $j = 1, 2, \ldots, k$ hold.

The completeness of the system of all eigen and associated functions of L is equivalent to the completeness of the system of all root functions of this operator.

If, for the operator L with dense domain $D(L)$ in H, the inequality $\Im(Ly, y) \geq 0$ $(y \in D(L))$ holds, then L is called dissipative.

Theorem 2.1 ([19]) *Let L be an invertible operator. Then, $-L$ is dissipative if and only if the inverse operator L^{-1} of L is dissipative.*

Lidskiĭ's Theorem ([13], p. 231) *If the dissipative operator L is the nuclear operator, then its system of root functions is complete in H.*

A linear bounded operator A defined on the seperable Hilbert space H is said to be of trace class (nuclear) if the series

$$\sum_k (Ae_k, e_k)$$

converges and has the same value in any orthonormal basis $\{e_k\}$ of H. The sum

$$TrA = \sum_k (Ae_k, e_k)$$

is called the trace of A.

The kernel $G(s,t),\ s,t \in \mathbb{R}$, of the integral operator K on $L^2(\mathbb{R})$

$$Kf = \int_{\mathbb{R}} G(s,t) f(s) ds,\ \ f \in L^2(\mathbb{R}),$$

is a Hilbert–Schmidt kernel if $|G(s,t)|^2$ is integrable on $\mathbb{R}^2$, i.e.,

$$\int_{\mathbb{R}^2} |G(s,t)|^2 \, dsdt < \infty.$$

A Hilbert–Schmidt kernel which is measurable and such that $G(s,s)$ is integrable on $\mathbb{R}$ is called a trace-class kernel [25, p. 79], [26, p. 526].

Integral operator with trace class kernel is nuclear.

3 Statement of the problem

Let η be the differential expression defined by

$$\eta(y) = \frac{1}{w(x)} \left[-(p(x)y')' + q(x)y \right],\ x \in \Lambda := \bigcup_{k=1}^{n+1} \Lambda_k,$$

where $\Lambda_k = (c_{k-1}, c_k)$ and $-\infty < c_0 < c_1 < \cdots < c_{n+1} \leq \infty$. It is assumed that the points $c_0, c_1, \ldots, c_n$ are regular and c_{n+1} is singular for the differential expression $\eta,\ p, q$ and w are real-valued, Lebesgue measurable functions on $\Lambda,\ p^{-1}, q, w \in L^1_{loc}(\Lambda_k),\ k = 1, 2, \ldots, n+1$, and, $w(x) > 0$ for almost all x on Λ.

Let $L^2_w(\Lambda)$ be the Hilbert space consisting of all complex valued functions y such that $\int_\Lambda w(x) \, |y(x)|^2 \, dx < \infty$ with the inner product

$$(y, \chi) = \int_\Lambda w(x) y(x) \overline{\chi}(x) dx.$$

Let

$$D = \left\{ y \in L^2_w(\Lambda) : y, py' \in AC_{loc}(\Lambda_k), \eta(y) \in L^2_w(\Lambda) \right\},$$

where $AC_{loc}(\Lambda_k),\ k = 1, 2, \ldots, n+1$, denotes the set consisting of all locally absolutely continuous functions on Λ_k.

For arbitrary $y, \chi \in D$, Green's formula is

$$\int_{\Lambda} w(x)\eta(y)\overline{\chi}(x)dx - \int_{\Lambda} w(x)y(x)\overline{\eta(\chi)}dx = \sum_{k=1}^{n+1}[y,\chi]_{c_{k-1}+}^{c_k-},$$

where $[y,\chi]_{c_{k-1}+}^{c_k-} = [y,\chi]_{c_k-} - [y,\chi]_{c_{k-1}+}$, $[y,\chi]_x := y(x)\overline{\chi^{[1]}}(x) - y^{[1]}(x)\overline{\chi}(x)$ and $y^{[1]}$ denotes py'. Green's formula implies that at singular point c_{n+1} for all $y, \chi \in D$, the limit $[y,\chi]_{c_{n+1}} := [y,\chi]_{c_{n+1}-} = \lim_{x\to c_{n+1}-}[y,\chi]_x$ exists and is finite.

In this paper it is assumed that the functions p, q and w satisfy the Weyl's limit-circle case conditions at singular point c_{n+1}. Weyl's theory is well known and there are several sufficient conditions in which Weyl's limit-circle case holds for a differential expression [27–30].

Let us consider the solutions $\varphi(x,\lambda) = \{\varphi_1(x,\lambda), \varphi_2(x,\lambda), \ldots, \varphi_{n+1}(x,\lambda)\}$ and $\psi(x,\lambda) = \{\psi_1(x,\lambda), \psi_2(x,\lambda), \ldots, \psi_{n+1}(x,\lambda)\}$, where $\varphi_k(x,\lambda)$ and $\psi_k(x,\lambda)$ are the parts of the functions $\varphi(x,\lambda)$ and $\psi(x,\lambda)$, respectively, defined on the interval Λ_k $(k = 1, 2, \ldots, n+1)$, of the equation

$$-(p(x)y')' + q(x)y = \lambda w(x)y, \ x \in \Lambda, \tag{3.1}$$

where λ is some complex parameter, satisfying the conditions [5–9,22–24]

$$\begin{cases} \varphi_1(c_0,\lambda) = \cos\alpha, & \varphi_1^{[1]}(c_0,\lambda) = \sin\alpha, \\ \psi_1(c_0,\lambda) = -\sin\alpha, & \psi_1^{[1]}(c_0,\lambda) = \cos\alpha, \end{cases}$$

and

$$\begin{cases} \varphi_{m+1}(c_m+,\lambda) = \frac{1}{\gamma_m}\varphi_m(c_m-,\lambda), & \varphi_{m+1}^{[1]}(c_m+,\lambda) = \frac{1}{\gamma_m'}\varphi_m^{[1]}(c_m-,\lambda), \\ \psi_{m+1}(c_m+,\lambda) = \frac{1}{\gamma_m}\psi_m(c_m-,\lambda), & \psi_{m+1}^{[1]}(c_m+,\lambda) = \frac{1}{\gamma_m'}\psi_m^{[1]}(c_m-,\lambda), \end{cases}$$

where α, γ_m and γ_m' are some real numbers with $\gamma_m\gamma_m' > 0$ and $m = 1, 2, \ldots, n$. Since Weyl's limit-circle case holds at singular point c_{n+1} for η, the solutions $\varphi(x,\lambda)$ and $\psi(x,\lambda)$ $(x \in \Lambda)$ belong to $L_w^2(\Lambda)$.

Let $z(x) = \{z_1(x), z_2(x), \ldots, z_{n+1}(x)\}$ and $u(x) = \{u_1(x), u_2(x), \ldots, u_{n+1}(x)\}$ be the solutions of $\eta(y) = 0$ $(x \in \Lambda)$ satisfying the conditions

$$\begin{cases} z_1(c_0) = \cos\alpha, & z_1^{[1]}(c_0) = \sin\alpha, \\ u_1(c_0) = -\sin\alpha, & u_1^{[1]}(c_0) = \cos\alpha, \end{cases}$$

and

$$\begin{cases} z_{m+1}(c_m+) = \frac{1}{\gamma_m}z_m(c_m-), & z_{m+1}^{[1]}(c_m+) = \frac{1}{\gamma_m'}z_m^{[1]}(c_m-), \\ u_{m+1}(c_m+) = \frac{1}{\gamma_m}u_m(c_m-), & u_{m+1}^{[1]}(c_m+) = \frac{1}{\gamma_m'}u_m^{[1]}(c_m-), \end{cases}$$

where α, γ_m and γ'_m are some real numbers with $\gamma_m\gamma'_m > 0$ and $m = 1, 2, \ldots, n$. It is clear that $z(x) = \varphi(x, 0)$ $(x \in \Lambda)$ and $u(x) = \psi(x, 0)$ $(x \in \Lambda)$. Hence $z(x)$ and $u(x)$ belong to $L^2_w(\Lambda)$. Further they belong to D. This implies that for each $y \in D$, at singular point c_{n+1}, the values $[y, z]_{c_{n+1}}$ and $[y, u]_{c_{n+1}}$ exist and are finite.

It should be noted that $[y, \overline{\chi}]_x$ $(x \in \Lambda)$ denotes the Wronskian of the solutions $y = y(x, \lambda)$ and $\chi = \chi(x, \lambda)$ of (3.1).

For $y \in D$, let us consider the following boundary and impulsive conditions

$$y(c_0)\cos\alpha + y^{[1]}(c_0)\sin\alpha = 0, \tag{3.2}$$

$$[y, z]_{c_{n+1}} - h[y, u]_{c_{n+1}} = 0, \tag{3.3}$$

$$y(c_m-) = \gamma_m y(c_m+), \tag{3.4}$$

$$y^{[1]}(c_m-) = \gamma'_m y^{[1]}(c_m+), \tag{3.5}$$

where α, γ_m and γ'_m are real numbers with $\gamma_m\gamma'_m > 0$, $m = 1, 2, \ldots, n$, and h is some complex number such that $h = \Re h + i\Im h$ with $\Im h > 0$. The main aim of present paper is to investigate the spectral analysis of the problem (3.1)–(3.5).

It is better to note that for the values $\gamma_m = \gamma'_m = 1$ $(m = 1, 2, \ldots, n)$, the problem was investigated in [16] with $p(x) = w(x) = 1$, [17] with $w(x) = 1$, [20] with $p(x) = w(x) = 1$ and [21]. Further for $m = 1$, the problem was studied in [22,24] with $p(x) = w(x) = 1$ and [23] with $p(x) = 1$, $q(x) = \frac{\nu^2-\frac{1}{4}}{x^2} + q_1(x)$, where $q_1(x)$ is continuous function on some interval.

4 Completeness theorem

Let $H = \bigoplus_{k=1}^{n+1} H_k$, $H_k = L^2_{w_k}(\Lambda_k)$, be the Hilbert space with the inner product

$$\langle y, \chi\rangle_H = [(y_1, \chi_1)_{H_1}, (y_2, \chi_2)_{H_2}, \ldots, (y_{n+1}, \chi_{n+1})_{H_{n+1}}][1, \Upsilon_1, \ldots, \Upsilon_1\Upsilon_2\ldots\Upsilon_n]^T,$$

where $(y_k, \chi_k)_{H_k} = \int_{\Lambda_k} w_k(x)y_k(x)\overline{\chi_k}(x)dx$, $k = 1, 2, \ldots, n+1$, $y(x) = \{y_1(x), y_2(x), \ldots, y_{n+1}(x)\} \in H$, $\chi(x) = \{\chi_1(x), \chi_2(x), \ldots, \chi_{n+1}(x)\} \in H$, $w(x) = \{w_1(x), w_2(x), \ldots, w_{n+1}(x)\}$, $\Upsilon_m := \gamma_m\gamma'_m$, $m = 1, 2, \ldots, n$, and $[.]^T$ denotes the transpose of the matrix $[.]$.

Let $D(N)$ be the set of all functions $y \in H$ such that $y, y^{[1]}$ are locally absolutely continuous functions on all Λ_k, $k = 1, 2, \ldots, n+1$, satisfying $\eta(y) \in H$, $R_-(y) = 0$, $R_+(y) = 0$, $R_m(y) = 0$ and $R'_m(y) = 0$, where $R_-(y) = y(c_0)\cos\alpha + y^{[1]}(c_0)\sin\alpha$, $R_+(y) = [y, z]_{c_{n+1}} - h[y, u]_{c_{n+1}}$, $R_m(y) = y(c_m-) - \gamma_m y(c_m+)$ and $R'_m(y) = y^{[1]}(c_m-) - \gamma'_m y^{[1]}(c_m+)$, $m = 1, 2, \ldots, n$.

Let N be the operator defined on $D(N)$ as $Ny = \eta(y)$ $(x \in \Lambda)$. Then the IBVP (3.1)–(3.5) can be handled in H as

$$Ny = \lambda y, \ y \in D(N), \ x \in \Lambda.$$

It is clear that the eigenvalues and the root lineals of N and the IBVP (3.1)–(3.5) coincide.

Consider the solution $\upsilon(x) = \{\upsilon_1(x),\ \upsilon_2(x),\ \ldots,\ \upsilon_{n+1}(x)\}$ of $\eta(y) = 0\,(x \in \Lambda)$, where $\upsilon(x) = z(x) - hu(x)\,(x \in \Lambda)$ and $\upsilon_k(x) = z_k(x) - hu_k(x)\ (x \in \Lambda_k)\,(k = 1, 2, \ldots, n+1)$. It is easy to see that $\upsilon(x)$ satisfies the boundary-impulsive conditions (3.3)–(3.5). On the other hand $u(x)$ satisfies the boundary-impulsive conditions (3.2), (3.4), (3.5).

Let $\Delta_k = [u_k, \upsilon_k]_x\ (x \in \Lambda_k),\ k = 1, 2, \ldots, n+1$. Then the equalities

$$\Delta_1 = -1,\ \Delta_2 = -\frac{1}{\Upsilon_1}, \ldots, \Delta_{n+1} = -\frac{1}{\Upsilon_1\Upsilon_2\ldots\Upsilon_n} \tag{4.1}$$

hold [16–18,22–24].

From (4.1), for arbitrary $y, \chi \in D(N)$ the following equalities are obtained

$$\begin{aligned}
&[y_1, \chi_1]_x = [y_1, z_1]_x\,[\overline{\chi}_1, u_1]_x - [y_1, u_1]_x[\overline{\chi}_1, z_1]_x, \quad x \in \Lambda_1,\\
&[y_2, \chi_2]_x = \Upsilon_1\big\{[y_2, z_2]_x[\overline{\chi}_2, u_2]_x - [y_2, u_2]_x[\overline{\chi}_2, z_2]_x\big\}, \quad x \in \Lambda_2,\\
&\vdots\\
&\big[y_{n+1}, \chi_{n+1}\big]_x = \Upsilon_1\Upsilon_2\ldots\Upsilon_n\{[y_{n+1}, z_{n+1}]_x[\overline{\chi}_{n+1}, u_{n+1}]_x\\
&\quad -[y_{n+1}, u_{n+1}]_x[\overline{\chi}_{n+1}, z_{n+1}]_x\}, \quad x \in \Lambda_{n+1}.
\end{aligned} \tag{4.2}$$

Theorem 4.1 *The operator N is dissipative in H.*

Proof For $y \in D(N)$, a direct calculation gives

$$\langle Ny, y\rangle_H - \langle y, Ny\rangle_H = [y, y]_{c_0+}^{c_1-} + \Upsilon_1[y, y]_{c_1+}^{c_2-} + \cdots + \Upsilon_1\Upsilon_2\ldots\Upsilon_n[y, y]_{c_n+}^{c_{n+1}-}. \tag{4.3}$$

The conditions $R_-(y) = 0,\ R_m(y) = 0$ and $R'_m(y) = 0\,(m = 1, 2, \ldots, n)$ give

$$[y, y]_{c_0+} = 0, [y, y]_{c_1-} = \Upsilon_1[y, y]_{c_1+}, \ldots, [y, y]_{c_n-} = \Upsilon_n[y, y]_{c_{n+1}}. \tag{4.4}$$

Further form the formula (4.2) and the condition $R_+(y) = 0$, we get that

$$[y, y]_{c_{n+1}} = \Upsilon_1\Upsilon_2\ldots\Upsilon_n 2i\Im h\left|[y, u]_{c_{n+1}}\right|^2. \tag{4.5}$$

Substituting (4.4) and (4.5) in (4.3) we have

$$\Im\langle Ny, y\rangle_H = (\Upsilon_1\Upsilon_2\ldots\Upsilon_n)^2\,\Im h\left|[y, u]_{c_{n+1}}\right|^2 \tag{4.6}$$

and this completes the proof. □

Theorem 4.2 *The operator N has no real eigenvalue.*

Proof Let λ_0 be a real eigenvalue of N and $\psi = \psi(x, \lambda_0)$ $(x \in \Lambda)$ be the corresponding eigenfunction of λ_0. A direct calculation gives

$$\Im \langle N\psi, \psi\rangle_H = \Im \left(\lambda_0 \|\psi\|_H^2\right). \tag{4.7}$$

Since λ_0 is a real number, from (4.7) and (4.6) we get that $[\psi, u]_{c_{n+1}} = 0$. Hence from (3.3) we have $[\psi, z]_{c_{n+1}} = 0$.

Setting $\varphi = \varphi(x, \lambda_0)$ $(x \in \Lambda)$ and using (4.2) one gets that

$$\begin{aligned} 1 &= \Upsilon_1\Upsilon_2 \ldots \Upsilon_n[\varphi, \overline{\psi}]_{c_{n+1}} \\ &= (\Upsilon_1\Upsilon_2 \ldots \Upsilon_n)^2 \{[\varphi_{n+1}, z_{n+1}]_{c_{n+1}}[\psi_{n+1}, u_{n+1}]_{c_{n+1}} \\ &\quad -[\varphi_{n+1}, u_{n+1}]_{c_{n+1}}[\psi_{n+1}, z_{n+1}]_{c_{n+1}}\} = 0. \end{aligned}$$

This contradiction completes the proof. □

From Theorems 4.1 and 4.2, we get that all eigenvalues of N lie in the open upper half-plane.

In particular zero is not an eigenvalue of N.

Let us consider the functions

$$\zeta_1(x, \lambda) = [\psi_{n+1}(x, \lambda), z_{n+1}(x)]_x,\ \ \zeta_2(x, \lambda) = [\psi_{n+1}(x, \lambda), u_{n+1}(x)]_x.$$

If we set

$$\zeta(\lambda) := \zeta_1(c_{n+1}, \lambda) - h\zeta_2(c_{n+1}, \lambda),$$

then the zeros of $\zeta(\lambda)$ coincide with the eigenvalues of the operator N.

Theorem 4.3 *The function $\zeta(\lambda)$ is an entire function.*

Proof For the solution $y = y(x, \lambda) = \{y_1(x, \lambda), y_2(x, \lambda), \ldots, y_{n+1}(x, \lambda)\}$ of Eq. (3.1), it is possible to get that

$$y_{n+1} = \Upsilon_1\Upsilon_2 \ldots \Upsilon_n \left([y_{n+1}, u_{n+1}]_x z_{n+1} - [y_{n+1}, z_{n+1}]_x u_{n+1}\right),\ \ x \in \Lambda_{n+1}. \tag{4.8}$$

Let

$$\Psi_1(x, \lambda) = [y_{n+1}, z_{n+1}]_x,\ \ \Psi_2(x, \lambda) = [y_{n+1}, u_{n+1}]_x,\ \ x \in \Lambda_{n+1}. \tag{4.9}$$

Following [31], for $x \in \Lambda_{n+1}$ we have

$$\begin{aligned} \frac{\partial}{\partial x}\Psi_1(x, \lambda) &= \lambda y_{n+1}(x, \lambda) z_{n+1}(x) w_{n+1}(x), \\ \frac{\partial}{\partial x}\Psi_2(x, \lambda) &= \lambda y_{n+1}(x, \lambda) u_{n+1}(x) w_{n+1}(x). \end{aligned} \tag{4.10}$$

Substituting (4.8) in (4.10) and using (4.9) we get that

$$\frac{\partial}{\partial x}\Psi(x,\lambda)=\lambda A(x)\Psi(x,\lambda),\ x\in\Lambda_{n+1}, \tag{4.11}$$

where

$$\Psi(x,\lambda)=\begin{bmatrix}\Psi_1(x,\lambda)\\ \Psi_2(x,\lambda)\end{bmatrix},$$
$$A(x)=\begin{bmatrix}-\Upsilon_1\Upsilon_2\ldots\Upsilon_n z_{n+1}(x)u_{n+1}(x)w_{n+1}(x) & \Upsilon_1\Upsilon_2\ldots\Upsilon_n z_{n+1}^2(x)w_{n+1}(x)\\ -\Upsilon_1\Upsilon_2\ldots\Upsilon_n u_{n+1}^2(x)w_{n+1}(x) & \Upsilon_1\Upsilon_2\ldots\Upsilon_n z_{n+1}(x)u_{n+1}(x)w_{n+1}(x)\end{bmatrix}.$$

Since $z_{n+1},u_{n+1}\in L^2_{w_{n+1}}(\Lambda_{n+1})$, the elements of $A(x)$ are in $L^1(\Lambda_{n+1})$. It is known [27,32] that for fixed $d\in\Lambda_1$, the functions $\psi_1(d,\lambda)$ and $\psi_1^{[1]}(d,\lambda)$ are entire functions of λ of order $\frac{1}{2}$. From transmission conditions (3.4), (3.5), all $\psi_k(e,\lambda)$ and $\psi_k^{[1]}(e,\lambda)$, $e\in\Lambda_k$, $k=2,3,\ldots,n+1$, are entire functions of λ of order $\frac{1}{2}$ for fixed $e\in\Lambda_k$. Hence $\zeta_j(b,\lambda)$ $(j=1,2)$ are entire functions of λ of order $\frac{1}{2}$ for fixed b, $c_n\le b<c_{n+1}$.

Let $y(x,\lambda)=\psi(x,\lambda)$ $(x\in\Lambda)$. Then from (4.11) it is obtained that

$$\widetilde{\zeta}(x,\lambda)=\widetilde{\zeta}(b,\lambda)+\lambda\int_b^x A(t)\widetilde{\zeta}(t,\lambda)dt,\ x\in\Lambda_{n+1}, \tag{4.12}$$

where

$$\widetilde{\zeta}(x,\lambda)=\begin{bmatrix}\zeta_1(x,\lambda)\\ \zeta_2(x,\lambda)\end{bmatrix}.$$

Using Gronwall inequality, from (4.12) we arrive at

$$\left\|\widetilde{\zeta}(x,\lambda)\right\|\le\left\|\widetilde{\zeta}(b,\lambda)\right\|\exp\left(|\lambda|\int_b^x\|A(t)\|\,dt\right),\ x\in\Lambda_{n+1}. \tag{4.13}$$

From (4.12) and (4.13) we get for $x\in\Lambda_{n+1}$ that

$$\begin{aligned}&\left\|\widetilde{\zeta}(c_{n+1},\lambda)-\widetilde{\zeta}(b,\lambda)\right\|\\ &\quad\le|\lambda|\left\|\widetilde{\zeta}(b,\lambda)\right\|\left(\int_b^{c_{n+1}}\|A(t)\|\,dt\right)\exp\left(|\lambda|\int_{c_n}^{c_{n+1}}\|A(t)\|\,dt\right).\end{aligned} \tag{4.14}$$

From (4.14) as $b\to c_{n+1}$, $\widetilde{\zeta}(b,\lambda)\to\widetilde{\zeta}(c_{n+1},\lambda)$, uniformly in λ in each compact set. Hence the proof is completed. □

Theorem 4.3 shows that all zeros of $\zeta(\lambda)$ (all eigenvalues of N) are discrete and possible limit points of these zeros (eigenvalues of N) can only occur at infinity.

For $y \in D(N)$, the equation $Ny = f(x)$ $(x \in \Lambda)$ is equivalent to the nonhomogeneous differential equation

$$\eta(y) = f(x), \ x \in \Lambda,$$

subject to the conditions

$$\begin{aligned} y(c_0)\cos\alpha + y^{[1]}(c_0)\sin\alpha &= 0, \\ [y, z]_{c_{n+1}} - h[y, u]_{c_{n+1}} &= 0, \quad \Im h > 0, \\ y(c_m-) &= \gamma_m y(c_m+), \\ y^{[1]}(c_m-) &= \gamma'_m y^{[1]}(c_m+), \end{aligned}$$

where $\gamma_m \gamma'_m > 0$, $m = 1, 2, \ldots, n$ and $f(x) = \{f_1(x), \ f_2(x), \ \ldots, \ f_{n+1}(x)\} \in L^2_w(\Lambda)$.

The general solution of the homogeneous differential equation can be represented as $y(x) = \{s_1 u_1(x) + p_1 v_1(x), \ldots, s_{n+1} u_{n+1}(x) + p_{n+1} v_{n+1}(x)\}$, where all s_k and p_k $(k = 1, 2, \ldots, n+1)$ are arbitrary constants [6,22–24].

By applying the standart method of variation of parameters the general solution is obtained as

$$\begin{aligned} y(x) = \Bigg\{ & u_1(x)\left[\Upsilon_1 \int_{c_1}^{c_2} f_2 v_2 w_2 d\xi + \cdots + \Upsilon_1\Upsilon_2\ldots\Upsilon_n \int_{c_n}^{c_{n+1}} f_{n+1} v_{n+1} w_{n+1} d\xi \right. \\ & \left. + \int_x^{c_1} f_1 v_1 w_1 d\xi \right] + v_1(x) \int_{c_0}^{x} f_1 u_1 w_1 d\xi, u_2(x)\left[\Upsilon_1\Upsilon_2 \int_{c_2}^{c_3} f_3 v_3 w_3 d\xi + \cdots \right. \\ & \left. + \Upsilon_1\Upsilon_2\ldots\Upsilon_n \int_{c_n}^{c_{n+1}} f_{n+1} v_{n+1} w_{n+1} d\xi + \Upsilon_1 \int_x^{c_2} f_2 v_2 w_2 d\xi \right] \\ & + v_2(x)\left[\int_{c_0}^{c_1} f_1 u_1 w_1 d\xi + \Upsilon_1 \int_{c_1}^{x} f_2 u_2 w_2 d\xi \right], \ldots, \\ & \Upsilon_1\Upsilon_2\ldots\Upsilon_n u_{n+1}(x) \int_x^{c_{n+1}} f_{n+1} v_{n+1} w_{n+1} d\xi + v_{n+1}(x) \\ & \times \left[\int_{c_0}^{c_1} f_1 u_1 w_1 d\xi + \Upsilon_1 \int_{c_1}^{c_2} f_2 u_2 w_2 d\xi + \cdots + \Upsilon_1\Upsilon_2\ldots\Upsilon_{n-1} \int_{c_{n-1}}^{c_n} f_n u_n w_n d\xi \right. \\ & \left. + \Upsilon_1\Upsilon_2\ldots\Upsilon_n \int_{c_n}^{x} f_{n+1} u_{n+1} w_{n+1} d\xi \right] \Bigg\}. \end{aligned}$$

Let us set

$$G(x,\xi)=\begin{cases}u(x)v(\xi),\ c_0\le x\le\xi\le c_{n+1};\,x,\xi\ne c_k;\,k=1,2,\ldots,n\\ u(\xi)v(x),\ c_0\le\xi\le x\le c_{n+1};\,x,\xi\ne c_k;\,k=1,2,\ldots,n\end{cases}\tag{4.15}$$

Then the general solution can be represented as

$$\begin{aligned}y(x)=&\left[(G(x,\xi),\overline{f}(\xi))_{H_1},(G(x,\xi),\overline{f}(\xi))_{H_2},\ldots,(G(x,\xi),\overline{f}(\xi))_{H_{n+1}}\right]\\&\times[1,\Upsilon_1,\ldots,\Upsilon_1\Upsilon_2\ldots\Upsilon_n]^T,\end{aligned}$$

that is,

$$y(x)=\left\langle G(x,\xi),\overline{f}(\xi)\right\rangle_H,$$

where $f\in L_w^2(\Lambda)$.

Let us consider the operator K defined by the formula

$$Kf=\left\langle G(x,\xi),\overline{f}(\xi)\right\rangle_H,\tag{4.16}$$

where $f\in L_w^2(\Lambda)$.

It is clear that $K=N^{-1}$. Consequently the root lineals of the operators K and N coincide. Hence the completeness of the system of all eigen and associated functions of K is equivalent to the completeness of those for N in H.

Since $u,v\in L_w^2(\Lambda)$ we obtain that $G(x,\xi)$ is a Hilbert–Schmidt kernel which is measurable and $G(x,x)$ is integrable on Λ. Hence K is of trace class. Let us consider the operator $-K$. Since N is a dissipative operator, hence from Theorem 2.1, $-K$ is also a dissipative operator. Thus all conditions are satisfied for the Lidskiĭ 's theorem. Hence we have;

Theorem 4.4 *The system of all root functions of $-K$ (also K)is complete in H.*

Since the completeness of the system of all root functions (eigen and associated functions) of K is equivalent to the completeness of those for N, and from all conclusions throughout the paper it is obtained;

Theorem 4.5 *All eigenvalues of the IBVP* (3.1)–(3.5) *lie in the open upper half-plane and they are purely discrete. The limit points of these eigenvalues can only occur at infinity. The system of all eigen and associated functions of the IBVP* (3.1)–(3.5) *is complete in $L_w^2(\Lambda)$.*

References

1. V. Lakshmikantham, D.D. Bainov, P.S. Simenov, *Theory of Impulsive Differential Equations, Series in Modern Applied Mathematics*, vol. 6 (World Scientific Publishing Co., Inc., Teaneck, NJ, 1989)
2. J. Walter, Regular eigenvalue problems with eigenvalue parameter in the boundary condition. Math. Z **133**, 301–312 (1973)

3. D.B. Hinton, An expansion theorem for an eigenvalue problem with eigenvalue parameter in the boundary condition. Quart. J. Math. Oxf. Ser. **30**, 33–42 (1979)
4. C.T. Fulton, Singular eigenvalue problems with eigenvalue parameter contained in the boundary conditions. Proc. R. Soc. Edinburgh Sect. A **87**, 1–34 (1980)
5. E. Tunç, O.Sh. Muhtarov, Fundamental solutions and eigenvalues of one boundary-value problem with transmission conditions. Appl. Math. Comp. **157**, 347–355 (2004)
6. Z. Akdoğan, M. Demirci, O.Sh. Mukhtarov, Green function of discontinuous boundary-value problem with transmission conditions. Math. Met. Appl. Sci. **30**, 1719–1738 (2007)
7. O.Sh. Mukhtarov, M. Kadakal, Some spectral properties of one Sturm–Liouville type problem with discontinous weight. Siberian Math. J. **46**(4), 681–694 (2005)
8. O.Sh. Mukhtarov, M. Kadakal, Discontinuous Sturm–Liouville problems containing eigenparameter in the boundary conditions. Acta Math. Sinica. Eng. Ser. **22**(5), 1519–1528 (2006)
9. J.-J. Ao, J. Sun, M.-Z. Zhang, Matrix representations of Sturm–Liouville problems with transmission conditions. CAMWA **63**(8), 1335–1348 (2012)
10. I.M. Glazman, *Direct Methods of Qualitative Spectral Analysis of Singular Differential Operators* (Israel Program for Scientific Translations, Jerusalem, 1965)
11. A.R. Sims, Secondary conditions for linear differential operators of the second order. J. Math. Mech. **6**, 247–285 (1957)
12. M.V. Keldysh, On the completeness of the eigenfunctions of some classes of non self-adjoint linear operators. Soviet Math. Dokl. **77**, 11–14 (1951)
13. I.C. Gohberg, M.G. Krein, *Introduction to the Theory of Linear Nonselfadjoint Operators* (American Mathematical Society, Providence, 1969)
14. M.A. Naimark, *Linear Differential Operators*, 2nd edn. (Nauka, Moscow, 1969); English transl. of 1st edn., Parts 1, 2 (Ungar, New York, 1967, 1968)
15. B.Sz. Nagy, C. Foiaş, *Harmonic Analysis of Operators on Hilbert Space* (Academia Kioda, Budapest, 1970)
16. G.Sh. Guseinov, H. Tuncay, The determinants of perturbation connected with a dissipative Sturm–Liouville operators. J. Math. Anal. Appl. **194**, 39–49 (1995)
17. E. Bairamov, A.M. Krall, Dissipative operators generated by the Sturm–Liouville expression in the Weyl limit circle case. J. Math. Anal. Appl. **254**, 178–190 (2001)
18. G. Guseinov, Completeness theorem for the dissipative Sturm–Liouville operator. Doga-Tr. J. Math. **17**, 48–54 (1993)
19. Z. Wang, H. Wu, Dissipative non-self-adjoint Sturm–Liouville operators and completeness of their eigenfunctions. J. Math. Anal. Appl. **394**, 1–12 (2012)
20. B. P. Allahverdiev, On dilation theory and spectral analysis of dissipative Schrödinger operators in Weyl's limit-circle case. Math. USSR Izvestiya **36**, 247–262 (1991)
21. B.P. Allahverdiev, A dissipative singular Sturm–Liouville problem with a spectral parameter in the boundary condition. J. Math. Anal. Appl. **316**, 510–524 (2006)
22. E. Bairamov, E. Ugurlu, The determinants of dissipative Sturm–Liouville operators with transmission conditions. Math. Comput. Model. **53**, 805–813 (2011)
23. E. Bairamov, E. Ugurlu, On the characteristic values of the real component of a dissipative boundary value transmission problem. Appl. Math. Comput. **218**, 9657–9663 (2012)
24. E. Bairamov, E. Ugurlu, Krein's theorems for a dissipative boundary value transmission problem. Complex Anal. Oper. Theory. doi:10.1007/s11785-011-0180-z
25. F. Smithies, *Integral Equations* (Cambridge University Press, Cambridge, 1958)
26. E. Prugovečki, *Quantum Mechanics in Hilbert Space*, 2nd edn. (Academic Press, New York, 1981)
27. E.C. Titchmarsh, *Eigenfunction Expansions Associated with Second Order Differential Equations, Part 1*, 2nd edn. (Oxford University Press, Oxford, 1962)
28. F.V. Atkinson, *Discrete and Continuous Boundary Problems* (Academic Press, New York, 1964)
29. B.J. Harris, Limit-circle criteria for second order differential expression. J. Math. Oxf. Ser. **35**(2), 415–427 (1984)
30. W.N. Everitt, I.W. Knowles, T.T. Read, Limit-point and limit-circle criteria for Sturm–Liouville equations with intermittently negative principal coefficient. Proc. R. Soc. Edinb. Sect. A. **103**, 215–228 (1986)
31. C.T. Fulton, Parametrization of Titchmarsh's $m(\lambda)$- functions in the limit circle case. Trans. Am. Math. Soc. **229**, 51–63 (1977)
32. A. Zettl, *Sturm-Liouville Theory, Mathematical Surveys and Monographs*, vol. 121 (American Mathematical Society, Providence, RI, 2005)